

# APPROXIMATIONS OF PSEUDO-DIFFERENTIAL FLOWS

BENJAMIN TEXIER

**ABSTRACT.** Given a classical symbol  $M$  of order zero, and associated semiclassical operators  $\text{op}_\varepsilon(M)$ , we prove that the flow of  $\text{op}_\varepsilon(M)$  is well approximated, in time  $O(|\ln \varepsilon|)$ , by a pseudo-differential operator, the symbol of which is the flow  $\exp(tM)$  of the symbol  $M$ . A similar result holds for non-autonomous equations, associated with time-dependent families of symbols  $M(t)$ . This result was already used, by the author and co-authors, to give a stability criterion for high-frequency WKB approximations, and to prove a strong Lax-Mizohata theorem. We give here two further applications: sharp semigroup bounds, implying nonlinear instability under the assumption of spectral instability at the symbolic level, and a new proof of sharp Gårding inequalities.

## 1. INTRODUCTION

Consider a family  $\text{op}_\varepsilon(M)$  of semiclassical pseudo-differential operators associated with a matrix-valued classical symbol  $M$  of order zero: that is  $M(x, \xi) \in \mathbb{C}^{N \times N}$ , for  $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ , satisfying the uniform bounds

$$(1.1) \quad |\partial_x^\alpha \partial_\xi^\beta M(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|^2)^{-|\beta|/2}, \quad C_{\alpha\beta} > 0, \quad \alpha, \beta \in \mathbb{N}^d,$$

and the associated family of operators defined on the Schwartz class by

$$(1.2) \quad (\text{op}_\varepsilon(M)u)(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} M(x, \varepsilon \xi) \hat{u}(\xi) d\xi, \quad \varepsilon > 0.$$

By the Calderón-Vaillancourt theorem, for all  $\varepsilon > 0$ ,  $\text{op}_\varepsilon(M)$  extends to a linear bounded operator  $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ . Denote  $\exp(\text{top}_\varepsilon(M))$  the flow of the ordinary differential equation

$$(1.3) \quad \partial_t u = \text{op}_\varepsilon(M)u,$$

known to exist and be global by the Cauchy-Lipschitz theorem, so that  $\exp(\text{top}_\varepsilon(M))u_0$  denotes the unique solution to (1.3) with value  $u_0 \in L^2(\mathbb{R}^d)$  at  $t = 0$ .

We show here that, in time  $O(|\ln \varepsilon|)$ , there holds the approximation

$$\exp(\text{top}_\varepsilon(M)) \simeq \text{op}_\varepsilon(\exp(tM)),$$

made precise in the Approximation Lemma 2.1 below.

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More generally, given a bounded family  $(M(t))_{t \in \mathbb{R}}$  in the space of symbols of order zero, we show that the solution to the initial value-problem

$$(1.4) \quad \partial_t u = \text{op}_\varepsilon(M(t))u, \quad u(0) = u_0 \in L^2(\mathbb{R}^d),$$

is well approximated, in time  $O(|\ln \varepsilon|)$ , by  $\text{op}_\varepsilon(S(0; t))u_0$ , where  $S$  is the solution operator for  $M(t)$ , defined by

$$(1.5) \quad \partial_t S(\tau; t) = M(t)S(\tau; t), \quad S(\tau; \tau) \equiv \text{Id}.$$

In other words, we are approximating solution operators to a class of ordinary differential equations in infinite dimensions (typically,  $L^2$ ) by pseudo-differential operators, the symbols of which are solution operators to ordinary differential equations in *finite* dimensions (typically,  $\mathbb{C}^{N \times N}$ )<sup>1</sup>.

This reduction to finite dimensions has applications in particular to *stability* problems, since spectra of variable-coefficient (pseudo)-differential operators are typically difficult to describe, while the spectra of their symbols, being spectra of families of matrices, are at least theoretically computable. Indeed, the Approximation Lemma was already used by the author and co-authors:

- in [11], we proved that for large-amplitude high-frequency WKB solutions to semilinear hyperbolic systems, stability is generically equivalent to preservation of hyperbolicity around resonant frequencies. The verification of this stability criterion involves only computation of spectra and eigenprojectors in finite dimensions. This result applies in particular to instabilities in coupled Klein-Gordon systems and to the Raman and Brillouin instabilities.

- In [10], we proved a strong Lax-Mizohata theorem stating that even a weak defect of hyperbolicity implies ill-posedness for systems of first-order partial differential equations, extending work of Métivier [12].

We give here two further applications:

- in Theorem 3.1, Section 3, sharp lower and upper bounds are proved for the solution operator to (1.3); in line with the above comment following equation (1.5), we note that we dispense here with any consideration of infinite-dimensional spectra of linear (pseudo)-differential operator, and derive growth estimates based solely on consideration of spectra of matrices (symbols). In Section 3.1, we observe that the bounds of Theorem 3.1 are typically sharper than bounds derived from Gårding's inequality, and in Section 3.2 we use Theorem 3.1 to prove a nonlinear instability result.

- In Section 4, we give a new proof of sharp Gårding inequalities with gain of  $\theta$  derivatives, for  $0 < \theta < 1$ , based on the Approximation Lemma 2.1. This somehow completes the comparison, initiated in Section 3, of Lemma 2.1 with Gårding's inequality.

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<sup>1</sup>The assumption that  $M$  be order zero is crucial for our purposes. Indeed, for the exponential  $e^M$  of a classical symbol  $M$  to be a symbol,  $M$  must belong to  $S^0$ . We could, however, do without the semiclassical quantization in (1.3) and (1.4). Indeed, in Section 4, we prove an Approximation Lemma for symbols in Weyl quantization; powers of  $\varepsilon$  are there replaced with gains in the orders of the operators.

## 2. THE APPROXIMATION LEMMA

Let  $M(t)$  be a bounded family in  $S^0$ , meaning a family of smooth maps  $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow M(t, x, \xi) \in \mathbb{C}^{N \times N}$ , such that the bounds (1.1) hold uniformly in  $(t, x, \xi)$ . Consider the associated ordinary differential equations

$$(2.1) \quad \begin{cases} \partial_t u = \text{op}_\varepsilon(M)u + f, \\ u(0) = u_0. \end{cases}$$

where  $\text{op}_\varepsilon(M)$  is defined in (1.2). In (2.1), the datum  $u_0$  belongs to  $H^s$ , and the source  $f$  is given in  $C^0([0, T] \ln \varepsilon, H^s(\mathbb{R}^d))$ , for some  $s \in \mathbb{R}$ ,  $T > 0$ .

Let  $S(\tau; t)$  be the (finite-dimensional) solution operator associated with  $M(t)$ , that is the family of solutions to the ordinary differential equations in  $\mathbb{C}^{N \times N}$ :

$$\partial_t S(\tau; t) = M(t)S(\tau; t), \quad S(\tau; \tau) \equiv \text{Id}.$$

By how much does  $\text{op}_\varepsilon(S(0; t))$  fail to be the operator solution to (2.1)? By composition of operators in semiclassical quantization, there holds

$$(2.2) \quad \text{op}_\varepsilon(\partial_t S) = \text{op}_\varepsilon(MS) = \text{op}_\varepsilon(M)\text{op}_\varepsilon(S) - \underbrace{\varepsilon \text{op}_\varepsilon(M\sharp S) - \varepsilon^2(\dots)}_{\text{error term}},$$

where  $\sharp$  denotes the bilinear map  $\sigma_1 \sharp \sigma_2 := \sum_{|\alpha|=1} -i \partial_\xi^\alpha \sigma_1 \partial_x^\alpha \sigma_2$ . Classical results on pseudo-

differential operators are recalled in the Appendix (Section 5); in particular a precise estimate for the error in (2.2) is given in (5.6)-(5.7).

We see in (2.2) that the leading term in the error is presumably  $\varepsilon \text{op}_\varepsilon(M\sharp S)$ , which, in times  $O(|\log \varepsilon|)$ , may be catastrophically large. Indeed, there holds, by Gronwall's lemma, the bound  $|S(\tau; t)| \leq e^{\gamma(t-\tau)}$ , where  $\gamma := |M|_{L^\infty}$ . This implies, via the representation

$$\partial_x^\alpha \partial_\xi^\beta S(\tau; t) = \int_\tau^t S(t'; t) [\partial_x^\alpha \partial_\xi^\beta, M(t')] S(\tau; t') dt',$$

the bounds

$$(2.3) \quad \langle \xi \rangle^{|\beta|} |\partial_x^\alpha \partial_\xi^\beta S| \lesssim |\log \varepsilon|^* e^{\gamma(t-\tau)},$$

where  $|\log \varepsilon|^* = |\log \varepsilon|^{N^*(\alpha)}$ , for some  $N^*(\alpha) \in \mathbb{N}$ , and  $\lesssim$  means inequality up to a multiplicative constant, depending on  $\alpha$ ,  $M$  and  $T$  but *not* on  $(\varepsilon, \tau, t)$ . Thus  $\varepsilon |M\sharp S| \lesssim \varepsilon |\log \varepsilon|^* e^{\gamma t}$ , and the upper bound is very large in time  $O(|\log \varepsilon|)$ , in spite of the  $\varepsilon$  prefactor.

We then introduce a first-order corrector  $S_1$ , defined by

$$\partial_t S_1 = MS_1 + M\sharp S, \quad S_1(\tau; \tau) = 0,$$

so that

$$(2.4) \quad S_1(\tau; t) = \int_\tau^t S(s; t) M(s) \sharp S(\tau; s) ds.$$

In particular,  $S_1 \in S^{-1}$ , and in time  $O(|\log \varepsilon|)$  the corrector  $S_1$  and its derivatives are growing at most at exponential rate  $\gamma$ , no faster than  $S$ , up to a prefactor of the form

$|\ln \varepsilon|^*$ , precisely:

$$(2.5) \quad \langle \xi \rangle^{-1+|\beta|} |\partial_x^\alpha \partial_\xi^\beta S_1| \lesssim |\ln \varepsilon|^* e^{\gamma(t-\tau)}.$$

The symbol  $S_0 + \varepsilon S_1$  is a candidate for a better approximation of the symbol of the solution operator, in that it satisfies

$$(2.6) \quad \partial_t \text{op}_\varepsilon(S_0 + \varepsilon S_1) = \text{op}_\varepsilon(M) \text{op}_\varepsilon(S_0 + \varepsilon S_1) - \varepsilon^2(\dots).$$

In the above error  $O(\varepsilon^2)$ , the leading term involves symbols like  $M \sharp S_1$ , which is not growing faster than  $S$ . Thus the error in (2.6) is truly smaller than the error in (2.2): the net gain is a power of  $\varepsilon$ , modulo possibly large, and essentially irrelevant, powers of  $|\log \varepsilon|$ .

Iterating this procedure, we define  $(S_q)_{1 \leq q \leq q_0}$ , for  $q_0 := [\gamma T] + 1$ , as the solution to the triangular system of linear ordinary differential equations

$$(2.7) \quad \partial_t S_q = M S_q + \sum_{\substack{q_1+q_2=q \\ 0 < q_1}} M \sharp_{q_1} S_{q_2}, \quad S_q(\tau; \tau) = 0, \quad 1 \leq q \leq q_0 = [\gamma T] + 1,$$

where the bilinear map  $\sharp_q$  is defined by

$$(2.8) \quad a_1 \sharp_q a_2 := \sum_{|\alpha|=q} \frac{(-i)^{|\alpha|}}{|\alpha|!} \partial_\xi^\alpha a_1 \partial_x^\alpha a_2.$$

From (2.7), we see that  $S_q$  satisfies bounds

$$(2.9) \quad \langle \xi \rangle^{-q+|\beta|} |\partial_x^\alpha \partial_\xi^\beta S_q(\tau; t)| \lesssim |\ln \varepsilon|^* e^{\gamma(t-\tau)}.$$

The approximate solution operator is defined as

$$(2.10) \quad \Sigma = S + \sum_{1 \leq q \leq q_0} \varepsilon^q S_q.$$

**Lemma 2.1** (Approximation Lemma). *The operator  $\text{op}_\varepsilon(\Sigma)$  is an approximate solution operator for the differential equation (2.1), in that it satisfies*

$$(2.11) \quad \partial_t \text{op}_\varepsilon(\Sigma) = \text{op}_\varepsilon(M) \text{op}_\varepsilon(\Sigma) + \varepsilon \text{op}_\varepsilon(\rho),$$

with  $\rho$  such that, for all  $u \in H^{-q_0-1}(\mathbb{R}^d)$ ,

$$(2.12) \quad |\text{op}_\varepsilon(\rho(\tau; t))u|_{L^2} \lesssim |\log \varepsilon|^{N^*} \|u\|_{\varepsilon, -q_0-1},$$

uniformly in  $0 \leq \tau \leq t \leq T |\log \varepsilon|$ , for some  $N^* = N^*(q_0, N, d) \in \mathbb{N}$ .

Above,  $\|\cdot\|_{\varepsilon, s}$  denotes the semiclassical Sobolev norm  $\|u\|_{\varepsilon, s} := |(1 + |\varepsilon \xi|^2)^{s/2} \hat{u}|_{L^2(\mathbb{R}_\xi^d)}$ .

*Proof.* By composition of operators (see (5.6)-(5.7)), there holds for  $q \geq 0$ , denoting  $S_0 := S$ ,

$$(2.13) \quad \varepsilon^q \text{op}_\varepsilon(M) \text{op}_\varepsilon(S_q) = \varepsilon^q \text{op}_\varepsilon(M S_q) + \sum_{1 \leq q_1 \leq q_0 - q} \varepsilon^{q+q_1} \text{op}_\varepsilon(M \sharp_{q_1} S_q) + \varepsilon^{q_0+1} \text{op}_\varepsilon(\rho_q),$$

where  $\rho_q = R_{q_0-q+1}(M, S_q)$ , using notation introduced in (5.6), satisfies the bound

$$(2.14) \quad \varepsilon^{q_0+1} |\text{op}_\varepsilon(\rho_q(\tau; t))u|_{L^2} \lesssim \varepsilon^{q_0+1} |\log \varepsilon|^* e^{\gamma(t-\tau)} \|u\|_{\varepsilon, -q_0-1},$$

for all  $u \in H^{-q_0-1}$ , uniformly in  $0 \leq \tau \leq t \leq T|\log \varepsilon|$ . Let  $\rho := -\sum_{0 \leq q \leq q_0} \rho_q$ . Summing (2.13) over  $q$ , we obtain

$$(2.15) \quad \text{op}_\varepsilon(M)\text{op}_\varepsilon(\Sigma) = \text{op}_\varepsilon(M\Sigma) + \sum_{\substack{0 \leq q \leq q_0 \\ 1 \leq q_1 \leq q_0 - q}} \varepsilon^{q_1} \text{op}_\varepsilon(M_{\sharp q_1}^\# S_q) - \varepsilon \text{op}_\varepsilon(\rho).$$

Besides, by definition of the correctors (2.7), there holds

$$\partial_t \text{op}_\varepsilon(\Sigma) = \text{op}_\varepsilon(M\Sigma) + \sum_{\substack{0 \leq q_1 + q_2 \leq q_0 \\ 0 < q_1}} \varepsilon^{q_1 + q_2} \text{op}_\varepsilon(M_{\sharp q_1}^\# S_{q_2}),$$

and comparing with (2.15) we obtain identity (2.11). The remainder  $\rho$  satisfies (2.12), simply by summation of bounds (2.14), since by choice of  $q_0$  there holds  $\varepsilon^{q_0+1} e^{\gamma(t-\tau)} \leq \varepsilon$ .  $\square$

The Approximation Lemma 2.1 leads to the representation theorem for (2.1):

**Theorem 2.2.** *For  $\varepsilon$  small enough, depending on  $d, N, M$  and  $T$ , the unique solution  $u \in C^0([0, T|\log \varepsilon|], L^2(\mathbb{R}^d))$  to the initial-value problem (2.1) is given by*

$$(2.16) \quad u = \text{op}_\varepsilon(\Sigma(0; t))u_0 + \int_0^t \text{op}_\varepsilon(\Sigma(t'; t))(\text{Id} + \varepsilon R) \left( f - \varepsilon \text{op}_\varepsilon(\rho(0; \cdot))u_0 \right)(t') dt',$$

where  $\Sigma$  is defined in (2.10),  $\rho$  is the remainder in the Approximation Lemma 2.1, and  $R$  is linear bounded  $C^0([0, T|\log \varepsilon|], H^{-q_0-1}) \rightarrow C^0([0, T|\log \varepsilon|], L^2)$ , with bound

$$(2.17) \quad |(Rw)(t)|_{L^2} \lesssim |\log \varepsilon|^{N^*} \sup_{0 \leq t \leq T|\log \varepsilon|} \|w(t)\|_{\varepsilon, -q_0-1},$$

for some  $N^* > 0$ , depending on  $d, N, M, T$ , all  $w \in C^0 H^{-q_0-1}$ , uniformly in  $t \in [0, T|\log \varepsilon|]$ .

*Proof.* Let  $g \in C^0([0, T|\log \varepsilon|], L^2(\mathbb{R}^d))$ . By Lemma 2.1, the map  $u$  defined by

$$u := \text{op}_\varepsilon(\Sigma(0; t))u_0 + \int_0^t \text{op}_\varepsilon(\Sigma(t'; t))g(t') dt'$$

solves (2.1) if and only if, for all  $t$ ,

$$(2.18) \quad ((\text{Id} + \rho_0)g)(t) = f(t) - \varepsilon \text{op}_\varepsilon(\rho(0; t))u_0,$$

where  $\rho$  is the remainder introduced in Lemma 2.1 and  $\rho_0$  is the linear integral operator

$$\rho_0 : \quad v \in C^0 L^2 \rightarrow \left( t \rightarrow \varepsilon \int_0^t \text{op}_\varepsilon(\rho(t'; t))v(t') dt' \right) \in C^0 L^2.$$

By (2.12), there holds the uniform bound

$$(2.19) \quad \sup_{0 \leq t \leq T|\log \varepsilon|} |(\rho_0 u)(t)|_{L^2} \leq \varepsilon C |\log \varepsilon|^{1+N^*} \sup_{0 \leq t \leq T|\log \varepsilon|} \|u(t)\|_{\varepsilon, -q_0-1},$$

for some  $C = C(d, N, M, T) > 0$ . This implies that, for  $\varepsilon$  small enough, depending on  $d, N, M, T$ , the operator  $\text{Id} + \rho_0$  is invertible in the Banach algebra  $\mathcal{L}(C^0([0, T|\log \varepsilon|], L^2))$ . As a consequence, we can solve (2.18) in  $C^0 L^2$ , and obtain the representation formula (2.16), in which the remainder  $\varepsilon R := (\text{Id} + \rho_0)^{-1} - \text{Id}$  satisfies (2.17), by (2.19).  $\square$

**Remark 2.3.** *It may be useful to know exactly how small  $\varepsilon$  needs to be in Theorem 2.2. A look at the proofs of Lemma 2.1 and Theorem 2.2 shows that we need  $0 < \varepsilon < 1$  to be small enough so that  $\text{Id} + \rho_0$  be invertible. According to bound (2.19), this is implied by  $C\varepsilon|\ln \varepsilon|^{N^*} < 1$ . Bound (2.19) derives from bound (2.9) for the correctors. There the implicit constant depends on  $\|M\|_{q_0+C(d)}$ , for some  $C(d)$  depending only on  $d$ , where notation  $\|\cdot\|_r$  for symbol norms is introduced in (5.1), and the implicit exponent depends on  $q_0$  and  $N$ . Thus Theorem 2.2 applies as soon as*

$$(2.20) \quad \varepsilon|\ln \varepsilon|^{N^*} < C_0\|M\|_{\gamma T+C(d)}^{-1},$$

with positive constants  $C_0 = C_0(d, N)$ ,  $C(d)$ ,  $N^* = N^*(\gamma T, N, d)$ . In particular, the order of regularity required for the symbol, namely  $\gamma T + C(d)$ , is a function of its  $L^\infty$  norm  $\gamma$ .

### 3. APPLICATION: SHARP SEMIGROUP BOUNDS

The results of Section 2 translate into sharp semigroup bounds. Here as in Section 2,  $M$  is a symbol with values in  $\mathbb{C}^{N \times N}$ . We denote  $\sigma(M(x, \xi))$  the spectrum of matrix  $M(x, \xi)$ .

**Theorem 3.1.** *Given  $M \in S^0$  and  $T > 0$ , there holds for  $\varepsilon$  small enough the upper bound*

$$(3.1) \quad \left| \exp(\text{top}_\varepsilon(M)) \right|_{L^2 \rightarrow L^2} \leq C|\ln \varepsilon|^{N^*} \exp\left(t \sup_{x, \xi} \Re \sigma(M)\right),$$

uniformly in  $t \in [0, T|\log \varepsilon|]$ , for some  $N^* > 0$ ,  $C > 0$ , depending on  $d, N, M$ , and  $T$ . Besides, if  $\sup_{x, \xi} \Re \sigma(M)$  is attained, then for  $\varepsilon$  small enough we can find  $u_\varepsilon$  on the unit sphere of  $L^2$  such that, for some  $x_0 \in \mathbb{R}^d$ , there holds

$$(3.2) \quad C|\ln \varepsilon|^{-d} \exp\left(t \sup_{x, \xi} \Re \sigma(M)\right) \leq \left| \exp(\text{top}_\varepsilon(M))u_\varepsilon \right|_{L^2(B(x_0, |\ln \varepsilon|^{-1}))},$$

for some  $C > 0$  depending on  $d, N, M$  and  $T$ , uniformly in  $t \in [0, T|\log \varepsilon|]$ .

The bounds of Theorem 3.1 are sharp in the sense that the lower growth rate is equal to the upper growth rate. Note also that the above bounds coincide with sharp elementary bounds in the case of a symbol  $M = M(x)$  that is independent of  $\xi$ .

The proof below, and Remark 2.3 above, show that in order for (3.1) and (3.2) to hold, we need  $\varepsilon$  to satisfy a bound of the form  $\varepsilon|\ln \varepsilon|^{N^*(M, T)} < C(M, T)$ .

We discuss in Remark 3.2 how the condition that  $\sup_{x, \xi} \Re \sigma(M)$  be attained can be relaxed without any loss on the lower rate of growth.

*Proof.* By Lemma 2.1 and Theorem 2.2, there holds the bound, for  $\varepsilon$  small enough,

$$(3.3) \quad \left| \exp(\text{top}_\varepsilon(M)) \right|_{L^2 \rightarrow L^2} \leq |\text{op}_\varepsilon(\Sigma(0; t))|_{L^2 \rightarrow L^2} + \varepsilon|\ln \varepsilon|^* \int_0^t |\text{op}_\varepsilon(\Sigma(t'; t))|_{L^2 \rightarrow L^2} dt',$$

where  $\Sigma$  is the approximate solution operator defined in (2.10). We note here that in the bounds (2.3) and (2.5) for  $S$  and  $S_1$ , and similarly in the bounds for the higher-order correctors  $S_q$ , we can take  $\gamma = \sup_{x, \xi} \Re \sigma(M)$ . Indeed, there holds  $|e^{tM}| \lesssim (1+t)^{N-1}e^{t\gamma}$ , where  $N$  is the size of matrix  $M$ . By the Calderón-Vaillancourt theorem (bound (5.4) in Section 5), this implies for  $0 \leq t' \leq t \leq T|\ln \varepsilon|$  the bound

$$(3.4) \quad |\text{op}_\varepsilon(\Sigma(t'; t))|_{L^2 \rightarrow L^2} \leq C|\log \varepsilon|^{N^*} e^{(t-t')\gamma},$$

and using (3.4) in (3.3) we find (3.1). In the above upper bound, the constant  $C$  depends on a  $T$ -dependent norm of  $M$ , as discussed in Remark 2.3 above.

We turn to a proof of the lower bound (3.2). Let  $u_\varepsilon(x) := e^{ix \cdot \xi_0 / \varepsilon} \theta(x)$ , where  $\xi_0 \in \mathbb{R}^d$  and  $\theta \in C_c^\infty(\mathbb{R}^d)$ ,  $|\theta|_{L^2} = 1$ , will be appropriately chosen below. There holds

$$\text{op}_\varepsilon(e^{tM})u_\varepsilon = e^{ix \cdot \xi_0 / \varepsilon} \int_{\mathbb{R}^d} e^{ix \cdot \xi} e^{tM(x, \xi_0 + \varepsilon \xi)} \hat{\theta}(\xi) d\xi = e^{tM(x, \xi_0)} u_\varepsilon + \varepsilon v_\delta,$$

where

$$|v_\delta|_{L^2} := \left| \int_0^1 \int_{\mathbb{R}^d} e^{ix \cdot \xi} (\partial_\xi e^{tM})(x, \xi_0 + \varepsilon \tau \xi) \widehat{\partial_x \theta}(\xi) d\xi d\tau \right|_{L^2} \lesssim |\log \varepsilon|^* e^{t\gamma} |\partial_x \theta|_{L^2}.$$

Thus the task ahead is to find a bound from below for the family of vectors  $e^{tM(x, \xi_0)} \theta(x)$ . Since  $\gamma$  is attained, we can find  $(x_0, \xi_0) \in \mathbb{R}^{2d}$  such that  $\sigma(M(x_0, \xi_0)) \ni \lambda$ , with  $\Re \lambda = \gamma$ . The eigenvalue  $\lambda$  belongs to a continuous branch of eigenvalues  $\lambda(x)$  of  $M(x, \xi_0)$ , with  $\lambda(x_0) = \lambda$ . Let  $P(x)$  be the eigenprojector onto the generalized eigenspace associated with  $\lambda(x)$ , parallel to the sum of the other characteristic eigenspaces. The projector  $P$  is smooth in  $x$ , locally around  $x_0$ . We may assume that  $\theta \in \text{Ran } P$ . Thus there holds

$$e^{tM(x, \xi_0)} \theta(x) = e^{tM(x, \xi_0)} P(x) \theta(x) = e^{t\lambda(x)} \tilde{\theta}(t, x),$$

where

$$\tilde{\theta}(t, x) = \sum_{0 \leq j \leq r-1} t^j (M(x, \xi_0) - \lambda(x))^j \theta(x),$$

for some  $1 \leq r \leq N-1$  corresponding to the size of the Jordan block associated with  $\lambda$ . We may further choose  $\theta$  such that  $(M(x_0, \xi_0) - \lambda)^{r-1} \theta(x_0) \neq 0$ . The eigenvalue  $\lambda$  may not be smooth, but enjoys a Puiseux expansion in every direction, with Puiseux exponent  $1/r$ . Thus there holds

$$\Re \lambda(x) \geq \Re \lambda - C \zeta^{1/r}, \quad \text{for } |x - x_0| \leq \zeta,$$

with  $C > 0$  depending on  $M$ , implying, for the choice  $\zeta = (2C |\ln \varepsilon|)^{-r}$ , the bound

$$|e^{tM(x, \xi_0)} \theta(x)| \geq e^{t(\gamma - |\ln \varepsilon|^{-1})} |\tilde{\theta}(t, x)|, \quad |x - x_0| \leq \zeta.$$

It suffices to prove the lower bound (3.2) for  $t \geq 1$ . Then, the leading term in  $\tilde{\theta}$  is  $t^r (M - \lambda)^r \theta$ , with an  $L^2(B(x_0, |\ln \varepsilon|^{-1}))$  norm that is bounded from below by  $C_0 |\ln \varepsilon|^{-d}$ , where  $C_0$  is independent of  $\varepsilon$ . We obtained the lower bound

$$(3.5) \quad |\text{op}_\varepsilon(e^{tM})u_\varepsilon|_{L^2(B(x_0, \zeta))} \geq \frac{1}{2} C_0 |\ln \varepsilon|^{-d} e^{t\gamma} - \varepsilon C' |\log \varepsilon|^* e^{t\gamma},$$

where  $C' > 0$  is independent of  $\varepsilon, t$ .

From (3.5) we now deduce a lower bound for  $|\exp(\text{top}_\varepsilon(M))u_\varepsilon|$ , as follows. By Theorem 2.2, for  $\varepsilon$  small enough there holds

$$(3.6) \quad \begin{aligned} |\exp(\text{top}_\varepsilon(M))u_\varepsilon|_{L^2(B(x_0, \zeta))} &\geq |\text{op}_\varepsilon(\Sigma(0; t))u_\varepsilon|_{L^2(B(x_0, \zeta))} \\ &\quad - \varepsilon |\ln \varepsilon|^* \int_0^t |\text{op}_\varepsilon(\Sigma(t'; t))|_{L^2 \rightarrow L^2} dt'. \end{aligned}$$



Bound (2.9) shows that the correctors  $S_q$  do now grow faster than  $S_0 = e^{tM}$ . With the Calderón-Vaillancourt theorem, this implies

$$|\mathrm{op}_\varepsilon(\Sigma(0; t))u_\varepsilon|_{L^2(B(x_0, \zeta))} \geq |\mathrm{op}_\varepsilon(e^{tM})u_\varepsilon|_{L^2(B(x_0, \zeta))} - C'\varepsilon|\ln \varepsilon|^* e^{t\gamma}.$$

Thus (3.6) together with the upper bound (3.1) implies, for  $t \leq T|\ln \varepsilon|$  :

$$(3.7) \quad |\exp(\mathrm{top}_\varepsilon(M))u_\varepsilon|_{L^2(B(x_0, \zeta))} \geq |\mathrm{op}_\varepsilon(e^{tM})u_\varepsilon|_{L^2(B(x_0, \zeta))} - C'\varepsilon|\ln \varepsilon|^* e^{t\gamma}.$$

Combining (3.7) with (3.5), we find (3.2).  $\square$

**Remark 3.2.** *If  $\sup_{x, \xi} \Re \sigma(M)$  is not attained, for  $\delta > 0$  small enough, there exists  $(x(\delta), \xi(\delta))$  and  $\lambda(\delta) \in \sigma(M(x(\delta), \xi(\delta)))$  with  $\Re \lambda(\delta) > \gamma - \delta$ . Suppose that the family  $P(x(\delta), \xi(\delta))$  of generalized eigenprojectors associated with  $\lambda(\delta)$  is bounded in  $\delta$ , as  $\delta$  ranges in an open neighborhood of zero. This holds for instance if  $M$  is written in canonical Jordan block. Then  $P$  is independent of  $\delta$ .*

*Substituting this assumption on  $P$  for the assumption that  $\sup_{x, \xi} \Re \sigma(M)$  be attained, we may adapt the proof of Theorem 3.1 as follows.*

*For  $\delta = |\ln \varepsilon|^{-1}$ , we have  $x_0, \xi_0, \lambda$ , all  $\varepsilon$ -dependent, such that  $\Re \lambda = \gamma - |\ln \varepsilon|^{-1}$ , and  $\lambda \in \sigma(M(x_0, \xi_0))$ . For all  $\varepsilon$ , we may choose  $\theta \in C_c^\infty$ , with  $0 \leq \theta \leq 1$ , such that the vector  $(M(x_0, \xi_0) - \lambda)^{r-1}P(x_0, \xi_0)\theta(x_0)$  is unitary. By continuity, the vectors  $M(x, \xi_0) - \lambda(x)^{r-1}P(x, \xi_0)\theta(x)$  have norm greater than  $1/2$  as  $x$  ranges in a small neighborhood of  $x_0$ . Borrowing notation from the proof of Theorem 3.1, we thus find that there holds the lower bound*

$$|\tilde{\theta}|_{L^2(B(x_0, |\ln \varepsilon|^{-1}))} \geq t^{r-1}C_0|\ln \varepsilon|^{-d} - t^{r-2}C'_0|\ln \varepsilon|^{-d},$$

*where  $C'_0$  is finite and independent of  $\varepsilon$  by assumption on  $P$ . We conclude as in the proof of Theorem 3.1 that (3.2) holds.*

### 3.1. Comparison with the spectral mapping theorem and Gårding's inequality.

We argue here that the bounds of Theorem 3.1 are more useful than bounds derived from the spectral mapping theorem, and sharper than bounds derived from Gårding's inequality.

The spectral mapping theorem yields the upper bound

$$(3.8) \quad |\exp(\mathrm{top}_\varepsilon(M))|_{L^2 \rightarrow L^2} \lesssim C_0(\varepsilon, \delta) \exp(t(\delta + \gamma_\varepsilon)),$$

for any  $\delta > 0$ , all  $t \geq 0$ , with the growth rate

$$\gamma_\varepsilon := \sup \Re \sigma(\mathrm{op}_\varepsilon(M)),$$

and  $C_0(\varepsilon, \delta) := \max_{0 \leq s \leq \max(1, T(\delta))} |\exp(\mathrm{sop}_\varepsilon(M))|_{L^2 \rightarrow L^2}$ , for some  $T(\delta) > 0$ .

In this respect, Theorem 3.1 corresponds to a reduction to finite dimensions, as announced in the introduction. Indeed, the growth rate  $\gamma_\varepsilon$  involves the spectra of the  $L^2 \rightarrow L^2$  operators  $\mathrm{op}_\varepsilon(M)$ , while the upper bound (3.1) in Theorem 3.1 involves only the  $N \times N$  matrices  $M(x, \xi)$ . In particular,  $\gamma_\varepsilon$  might be very difficult to compute, while  $\sup \Re \sigma(M)$  is readily computable, at least in theory.

*Verification of (3.8):* first, by the semigroup property and Gelfand's formula, for any  $\eta > 0$ , for  $t \geq t(\eta)$ , there holds

$$(3.9) \quad |\exp(\mathrm{top}_\varepsilon(M))|_{L^2 \rightarrow L^2} \leq c_0 |\exp([t]\mathrm{op}_\varepsilon(M))|_{L^2 \rightarrow L^2} \leq c_0 \left( \rho(\exp \mathrm{op}_\varepsilon(M)) + \eta \right)^{[t]},$$



where  $\rho(\cdot)$  denotes spectral radius, and  $c_0(\varepsilon) = \max_{0 \leq s \leq 1} |\exp(\text{sop}_\varepsilon(M))|_{L^2 \rightarrow L^2}$ . Second, as a consequence of the spectral mapping theorem (see for instance [4], Lemma 3.13),

$$(3.10) \quad \rho(\exp \text{op}_\varepsilon(M)) \leq \exp \left( \sup \Re \sigma(\text{op}_\varepsilon(M)) \right).$$

Finally, given  $\delta > 0$ , for some  $\eta = \eta(\delta)$  bound (3.9) combined with (3.10) yields (3.8).

Another classical way to derive semigroup bounds is Gårding's inequality, which yields the upper bound

$$(3.11) \quad \left| \exp(\text{top}_\varepsilon(M)) \right|_{L^2 \rightarrow L^2} \lesssim \exp \left( t \left( \sup_{x, \xi} \sigma(\Re M) + \varepsilon C(M) \right) \right),$$

where  $\Re M$  is the hermitian matrix  $\Re M := (M + M^*)/2$ , and  $C(M) > 0$  can be expressed in terms of a norm  $\|M\|_{C(d)}$  of  $M$  (see Sections 4 and 5).

When  $M$  is hermitian, the bounds (3.11) and (3.1) coincide to first order.

Otherwise, bound (3.1) is typically strictly sharper than (3.11), as shown by example  $M = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}$ , for which  $\max \Re \sigma(M) = (a_{12}a_{21})^{1/2}$ , assuming  $a_{12}a_{21} > 0$ , and  $\max \sigma(\Re M) = (a_{12} + a_{21})/2$ , strictly greater than  $(a_{12}a_{21})^{1/2}$  if  $a_{12} \neq a_{21}$ , by the arithmetic-geometric inequality. In this example  $M$  is symmetrizable. An example with a Jordan block is given by  $M = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ , where  $a > 0$ , for which  $\max \Re \sigma(M) = a < \max \sigma(\Re M) = a + 1/2$ .

*Verification of (3.11):* let  $u \in L^2$ , and consider the solution  $v$  to  $\partial_t v = \text{op}_\varepsilon(M)v$  issued from  $v(0) = u$ . Denoting  $\bar{\gamma} := \sup_{x, \xi} \sigma(\Re M)$ , we let  $w(t) = e^{-t\bar{\gamma}}v(t)$ . Then,  $w$  solves  $\partial_t w = \text{op}_\varepsilon(M - \bar{\gamma})$ , so that  $(1/2)\partial_t |w|_{L^2}^2 = \Re(\text{op}_\varepsilon(M - \bar{\gamma})u, u)_{L^2}$ . Since  $\Re(M - \bar{\gamma}) \leq 0$ , we may apply Gårding's inequality (Theorem 4.1)<sup>2</sup>. This gives  $\partial_t |w|_{L^2}^2 \leq \varepsilon C(M)|w|_{L^2}^2$ , whence (3.11).

**3.2. Application to instability.** The semigroup bounds of Theorem 3.1 translate into instability results if the symbol  $M$  has unstable spectrum. Consider the situation of a semilinear equation

$$(3.12) \quad \partial_t u = \text{op}_\varepsilon(M)u + f(u),$$

where we assume, for  $s > d/2$ ,

$$(3.13) \quad \|f(u)\|_{\varepsilon, s} \leq C|u|_{L^\infty}\|u\|_{\varepsilon, s}, \quad \text{for some } C > 0, \text{ for all } u \in H^s.$$

Bounds (3.13) hold for example if  $f(u) = B(u, u)$ , with a bilinear  $B$ . The semiclassical Sobolev norm  $\|\cdot\|_{\varepsilon, s}$  is defined in (5.3).

**Theorem 3.3.** *If the spectrum of symbol  $M$  is unstable, meaning*

$$\text{sp } M(x, \xi) \cap \{z \in \mathbb{C}, \Re z > 0\} \neq \emptyset, \quad \text{for some } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

---

<sup>2</sup>Here we are using a semiclassical version of Gårding's inequality, in which the gain of one derivative in the remainder translates into a power of  $\varepsilon$ . As discussed in Section 4.5, our proof of Theorem 4.1 yields for matrix-valued symbols a gain of one-half of a derivative, implying an error in  $\varepsilon^{1/2}C(M)$  in (3.11).

and if  $\sup_{x,\xi} \Re \sigma(M)$  is attained, then for any  $K > 0$ , for some  $K' > 0$ , for  $\varepsilon$  small enough, some  $T_\star(\varepsilon) = O(|\ln \varepsilon|)$ , we can find a datum

$$u(0) \in C_c^\infty(\mathbb{R}^d), \quad \text{with} \quad |u(0)|_{L^\infty} = \varepsilon^K, \quad |u(0)|_{H^s} \lesssim \varepsilon^{K-s}, \quad \text{for all } s,$$

such that the solution  $u$  to (3.12)-(3.13) issued from  $u(0)$  belongs to  $L^\infty([0, T_\star(\varepsilon)], H^s(\mathbb{R}^d))$ , for all  $s > d/2$ , and satisfies

$$(3.14) \quad |u(T_\star(\varepsilon))|_{L^\infty(\mathbb{R}^d)} \geq |\ln \varepsilon|^{-K'}.$$

Thus we obtain a strong, albeit relative, instability under the mere assumption that the symbol has unstable spectrum. The instability is relative, in the sense that the deviation from the trivial solution depends on  $\varepsilon$ . The deviation, however, is strong: it is expressed in terms of an inverse power of  $\log$  (with an exponent,  $K'$ , which depends on  $M$  and  $K$ ), starting from an initial amplitude that is an arbitrarily large power of  $\varepsilon$ . In particular, the lower bound (3.14) implies  $|u(T_\star(\varepsilon))|_{L^\infty} \geq \varepsilon^\alpha$ , for any  $\alpha > 0$  and  $\varepsilon$  small enough. By comparison, less than optimal semigroup bounds would only imply a lower bound in  $\varepsilon^{\alpha(K)}$ , for some  $\alpha(K) > 0$ . We expand on this point after the proof of Theorem 3.3.

The result of Theorem 3.3 still holds if for the condition that  $\sup_{x,\xi} \Re \sigma(M)$  be attained we substitute the condition described in Remark 3.2, except for the  $H^s$  bound on the datum, which might not be  $O(\varepsilon^{K-s})$ , since the observation frequency  $\xi_0$  in this case depends on  $\varepsilon$ .

*Proof.* Let  $\gamma := \sup \Re \sigma(M) > 0$ . The limiting time is

$$(3.15) \quad T_\star(\varepsilon) := \frac{K}{\gamma} |\ln \varepsilon| - \frac{K_1}{\gamma} \ln |\ln \varepsilon|, \quad \text{for } K_1 > 0 \text{ to be chosen large enough.}$$

In particular,  $T_\star(\varepsilon) < \frac{K}{\gamma} |\ln \varepsilon|$ , and, following Remark 2.3, we may apply Theorem 3.1 as soon as  $\varepsilon$  is small enough, depending only on  $M$  and  $K$ . Let  $s > d/2$ . Local-in-time existence and uniqueness derive from the continuity of  $\text{op}_\varepsilon(M)$  as a linear operator from  $H^s$  to itself, and bound (3.13) on  $f$ . Via a classical continuation argument, existence and uniqueness up to  $T_\star(\varepsilon)$  thus follow from bounds over  $[0, T_\star(\varepsilon)]$ .

The solution  $u$  satisfies

$$u = \exp(\text{top}_\varepsilon(M))u(0) + \int_0^t \exp((t-t')\text{op}_\varepsilon(M))f(u(t'))dt'.$$

For  $\varepsilon$  small enough, we use Theorem 3.1. Continuity in  $\|\cdot\|_{\varepsilon,s}$  norms for the flow  $\exp(\text{top}_\varepsilon(M))$  follows from Theorem 2.2 and (5.8). Pointwise bounds are given by (5.13). Thus there holds, for the solution  $u$  to (3.12) issued from  $u(0)$ , the bounds

$$\|u\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^{N^*} \varepsilon^K e^{\gamma t} + |\ln \varepsilon|^{N^*} \int_0^t e^{\gamma(t-t')} |u|_{L^\infty} \|u\|_{\varepsilon,s} dt',$$

and

$$|u|_{L^\infty} \lesssim |\ln \varepsilon|^{N^*} \varepsilon^K e^{\gamma t} + |\ln \varepsilon|^{N^*} \int_0^t e^{\gamma(t-t')} |u|_{L^\infty}^2 \left(1 + |\ln \varepsilon| + |\ln |u|_{L^\infty}| + |\ln \|u\|_{\varepsilon,s}|\right) dt'.$$

Above, the exponent  $N^*$  depends only on  $M$  and  $K$ , as explained in Remark 2.3. The bound

$$|u|_{L^\infty} + \|u\|_{\varepsilon,s} \lesssim |\ln \varepsilon|^{N^*} \varepsilon^K e^{\gamma t}$$

is propagated so long as

$$|\ln \varepsilon|^{N^*} \varepsilon^K e^{\gamma t} \leq |\ln \varepsilon|^{-P},$$

with  $N_* + 1 < P$ , which in turn holds as soon as  $t \leq T_*(\varepsilon)$ , with the limiting time (3.15), if  $N_* + P < K_1$ .

Let now  $u(0, x) = \varepsilon^K e^{ix \cdot \xi_0 / \varepsilon} \theta(x)$ , with  $\theta \in C_c^\infty(\mathbb{R}^d)$ ,  $|\theta|_{L^2} \leq |\theta|_{L^\infty} = 1$ . By Theorem 3.1, for an appropriate choice of  $\xi_0$  and  $\theta$  there holds the lower bound, for  $t \leq T_*(\varepsilon)$ ,

$$|u(t)|_{L^2(B)} \geq C |\ln \varepsilon|^{-d} \varepsilon^K e^{\gamma t} - C' |\ln \varepsilon|^{N^*} \int_0^t e^{\gamma(t-t')} |u|_{L^\infty} |u|_{L^2} dt',$$

with  $B = B(x_0, |\ln \varepsilon|^{-1})$ , for some  $x_0 \in \mathbb{R}^d$ . With the above upper bounds, this gives

$$|u(t)|_{L^2(B)} \geq C |\ln \varepsilon|^{-d} e^{\gamma t} \varepsilon^K \left(1 - C'' |\ln \varepsilon|^{N^* - d - P}\right).$$

We now let  $P = N^* + d + 1$ , and  $K_1 = 2N^* + d + 1$ . Then, for  $\varepsilon$  small enough,

$$|u(t)|_{L^2(B)} \geq \frac{C}{2} |\ln \varepsilon|^{-(N-1)d} e^{\gamma t} \varepsilon^K,$$

implying (3.14), with  $K' = 2N^* + 2d + 1$ .  $\square$

Less than optimal semigroup bounds, such as given by Gårding's inequality, imply a much weaker form of instability for (3.12). Indeed, let  $\bar{\gamma}$  be an upper rate of exponential growth for  $\exp(\text{top}_\varepsilon(M))$ , let  $\gamma$  be a lower rate of growth, and assume  $\gamma < \bar{\gamma}$ . Then, disregarding powers of  $|\ln \varepsilon|$ , the goal is to compare the free solution to a Duhamel term of the form  $\int_0^t e^{\bar{\gamma}(t-t')} |u|_{L^\infty} |u|_{L^2} dt'$ . Existence is granted in time  $\frac{K}{\bar{\gamma}} |\ln \varepsilon|$ . The free solution grows in time like  $\varepsilon^K e^{t\gamma}$ . It dominates the Duhamel term only so long as  $|u|_{L^\infty} \leq \varepsilon^\alpha$  and  $t \leq \frac{\alpha}{\bar{\gamma} - \gamma} |\ln \varepsilon|$ . Thus the free solution is greater than  $\varepsilon^\alpha$  in time  $\frac{K - \alpha}{\bar{\gamma}} |\ln \varepsilon|$ , and we have a proof of a deviation  $|u|_{L^\infty} \geq \varepsilon^\alpha$  from  $|u(0)|_{L^\infty} = \varepsilon^K$  if  $\frac{K - \alpha}{\bar{\gamma}} < \frac{\alpha}{\bar{\gamma} - \gamma}$ . This is a Hölder type of deviation, in the sense that it expresses

$$(3.16) \quad \sup_{\substack{0 < \varepsilon < 1 \\ 0 \leq t \leq T_*(\varepsilon)}} \frac{|u(t)|_{L^\infty}}{|u(0)|_{L^\infty}^\beta} = \infty, \quad T_*(\varepsilon) = O(|\ln \varepsilon|), \quad \text{for } \beta > \frac{\bar{\gamma} - \gamma}{\gamma}.$$

If  $\bar{\gamma} < 2\gamma$ , the deviation estimate (3.16) indicates a lack of Hölder estimate for the flow of (3.12) in time  $O(|\ln \varepsilon|)$ . If  $\bar{\gamma} \geq 2\gamma$ , (3.16) is weaker than a lack of Lipschitz estimate. By comparison, Theorem 3.3 implies that there holds

$$\sup_{0 < \varepsilon < 1} \frac{|u(T_*(\varepsilon))|_{L^\infty}}{|u(0)|_{L^\infty}^\beta} = \infty, \quad T_*(\varepsilon) = O(|\ln \varepsilon|), \quad \text{for any } \beta > 0,$$

and also

$$\sup_{0 < \varepsilon < 1} \frac{\ln |u(T_*(\varepsilon))|_{L^\infty}}{\ln |u(0)|_{L^\infty}} \geq -K',$$

where  $K'$  depends on  $K$ , with  $|u(0)|_{L^\infty} = \varepsilon^K$ .

#### 4. APPLICATION: A NEW PROOF OF SHARP GÅRDING INEQUALITIES

We prove here the following Gårding inequalities:

**Theorem 4.1.** *For all  $m \in \mathbb{R}$ , all scalar symbol  $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$  such that  $\Re a \geq 0$ , for all  $0 < \theta < 1$ , some  $c > 0$ , there holds for all  $u \in H^m(\mathbb{R}^d)$  the lower bound*

$$\Re(\operatorname{op}(a)u, u)_{L^2} + c|u|_{H^{(m-\theta)/2}}^2 \geq 0.$$

The constant  $c$  depends on  $\theta$  and on a large number  $r(\theta)$  of derivatives of  $a$ , with  $c \rightarrow \infty$  and  $r \rightarrow \infty$  as  $\theta \rightarrow 1$ . This highlights two shortcomings of Theorem 4.1 and its proof: we do not handle the endpoint case  $\theta = 1$  corresponding to the classical Gårding inequality (first proved by Hörmander [6] for scalar symbols, and extended to systems by Lax and Nirenberg [8]), and we require a lot of smoothness for  $a$ .

Nonetheless our proof may have some interest in its own right. First, it completely differs from the classical proofs, which go either by reduction to the elliptic case (see for instance the proof of Theorem 4.32 in [14]), or by use of the Wick quantization (see for instance the proof of Theorem 1.1.26 in [9]). Second, it lends itself to partial extensions, in particular to the matrix case, as discussed in Section 4.5. Finally, it allows to view the Approximation Lemma 2.1 as a refinement of Gårding's inequality, in the sense that Lemma 2.1 implies Gårding (as shown by the proof below), and also implies stronger semigroup bounds than Gårding, as we saw in Section 3.

The proof of Theorem 4.1 is given in Sections 4.1 to 4.4. The key idea of the proof is the reformulation, in Section 4.2, of the Gårding inequality as an upper bound for the backwards flow of  $a^w$  (Weyl quantization). This is exploited in Section 4.3 where we approximate the flow of  $a^w$ , following the ideas of Section 2. Estimates conclude the proof in Section 4.4.

**4.1. First step: reductions.** We denote  $a^w$  the pseudo-differential operator in Weyl quantization (see definition (5.2) in the Appendix) with symbol  $a$ . There holds  $\operatorname{op}(a) = a^w + \operatorname{op}(\rho)$ , where  $\rho \in S^{-1}$ , with norms bounded by norms of  $a$ . In particular,  $(\operatorname{op}(\rho)u, u)_{L^2} \leq \|a\|_{C(d)}|u|_{H^{-(m-1)/2}}^2$ , for all  $u \in H^{m-1}$ . Thus we may switch to a Weyl quantization. The adjoint of  $a^w$  is  $(\bar{a})^w$ , so that  $\Re(a^w u, u)_{L^2} = ((\Re a)^w u, u)_{L^2}$ . Thus it suffices to handle the case  $a \in \mathbb{R}$ . The goal is now to prove

$$(4.1) \quad (a^w u, u)_{L^2} + c|u|_{H^{(m-\theta)/2}}^2 \geq 0, \quad \text{for any real } a \in S^m, \text{ some } c > 0, \text{ all } u \in H^m.$$

Let  $\Lambda = \operatorname{op}(\langle \cdot \rangle)$ , with  $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ , and  $a_0 := \langle \xi \rangle^{-m} a \in S_{1,0}^0$ . Consider the operator  $\Lambda^{m/2} a_0^w \Lambda^{m/2}$ . Its principal symbol is  $a$ . In Weyl quantization, its subprincipal symbol is

$$\frac{1}{2i} \{ \langle \xi \rangle^{m/2}, a_0 \langle \xi \rangle^{m/2} \} + \frac{1}{2i} \langle \xi \rangle^{m/2} \{ a_0, \langle \xi \rangle^{m/2} \} = 0.$$

Hence, by composition of operators (see (5.9) and (5.11)), there holds  $\Lambda^{m/2} a_0^w \Lambda^{m/2} = a^w + R_{m-2}$ , where  $(R_{m-2}w, w)_{L^2} \lesssim \|a\|_{C(d)}|w|_{H^{(m-2)/2}}^2$ , for all  $w \in H^{(m-2)/2}$ . Since  $\Lambda$  is  $L^2$ -self-adjoint,  $(a^w u, u)_{L^2} = (a_0^w \Lambda^{m/2} u, \Lambda^{m/2} u)_{L^2} + (R_{m-2}u, u)_{L^2}$ . From the above, it appears that it is sufficient to prove (4.1) in the case  $m = 0$ .

Let  $(\phi_j)_{j \geq 0}$  and  $(\psi_j^2)_{j \geq 0}$  be two dyadic Littlewood-Paley decompositions, such that  $\phi_j \equiv \phi_j \psi_j^2$ . Then there holds (this is Claim 2.5.24 in [9])

$$(4.2) \quad (a^w u, u)_{L^2} = \sum_{j \geq 0} ((\phi_j a)^w \psi_j(D)u, \psi_j(D)u)_{L^2} + c|u|_{H^{-1}}^2 \geq 0,$$

where  $c$  depends on norms of  $a$ . Thus it suffices to prove

$$(a_j^w u_j, u_j)_{L^2} + c2^{-j\theta}|u_j|_{L^2}^2 \geq 0, \quad a_j = \phi_j(\xi)a(x, \xi), \quad u_j := \psi_j(D)u,$$

for some  $c > 0$  independent of  $j$ . In particular, we may assume  $a_j \geq 2^{-j\theta}$ , for all  $j$ . We note moreover that low-frequency terms can be absorbed in the remainder, via

$$\sum_{0 \leq j \leq j_0} (a_j^w u_j, u_j)_{L^2} \lesssim \|a\|_0 2^{j_0\theta} |u|_{H^{-\theta/2}}^2,$$

a consequence of the  $L^2$  continuity of the  $a_j$  (see (5.5)). Finally, up to dividing by  $\|a\|_{C(\theta)}$ , where  $C(\theta)$  is large enough, we may assume that a large number of norms of  $a$  are bounded by 1.

In accordance with the above, in the rest of this proof a symbol  $a$  is given, such that

$$(4.3) \quad a \text{ is real, } a \in S^0, \quad a_j := \phi_j^2 a \geq 2^{-j\theta} \text{ for all } j, \quad \|a_j\|_{C(\theta)} \leq \|a\|_{C(\theta)} \leq 1,$$

with  $C(\theta)$  possibly large, and we undertake to find  $j_0 \in \mathbb{N}$  such that for all  $u \in L^2$ , all  $j \geq j_0$ ,

$$(4.4) \quad (a_j^w u_j, u_j)_{L^2} \geq 0, \quad u_j := \psi_j(D)u.$$

**4.2. Second step: reformulation in terms of the flow of  $a_j^w$ .** By the Calderón-Vaillancourt theorem (5.5), the operator  $a_j^w$  is linear bounded  $L^2 \rightarrow L^2$ . Let  $\Phi$  be the flow of  $a_j^w$  :  $\Phi(t) = \exp(ta_j^w)$ , meaning that for all  $w \in L^2$ , for all  $t \in \mathbb{R}$ ,  $\Phi(t)w$  is the unique solution in  $L^2$  to the initial-value problem

$$y' = a_j^w y, \quad y(0) = w.$$

We compute, for  $t \in \mathbb{R}$  and  $w \in L^2$ ,

$$(4.5) \quad \frac{1}{2} \frac{d}{dt} |\Phi(t)w|_{L^2}^2 = (a_j^w \Phi(t)w, \Phi(t)w)_{L^2},$$

and

$$(4.6) \quad \frac{1}{2} \frac{d}{dt} (a_j^w \Phi(t)w, \Phi(t)w)_{L^2} = 2|a_j^w \Phi(t)w|_{L^2}^2 \geq 0,$$

so that the right-hand side in (4.5) is a growing function of time. Integrating (4.5) from 0 to  $t$ , we find

$$|\Phi(t)w|_{L^2}^2 - |w|_{L^2}^2 = 2 \int_0^t (a_j^w \Phi(t')w, \Phi(t')w)_{L^2} dt',$$

hence with (4.6), the inequality

$$(4.7) \quad |\Phi(t)w|_{L^2}^2 - |w|_{L^2}^2 \leq 2t (a_j^w \Phi(t)w, \Phi(t)w)_{L^2}, \quad t \geq 0, w \in L^2.$$

For all  $t$ , the operator  $\Phi(t)$  is onto  $L^2$  (indeed, there holds  $\text{Id}_{\mathcal{L}(L^2)} = \Phi(t)\Phi(-t)$ ), so that, for  $u_j$  defined in (4.4), we can write  $u_j = \Phi(t)w$  with  $w = \Phi(-t)u_j$ , and (4.7) becomes

$$|u_j|_{L^2}^2 - |\Phi(-t)u_j|_{L^2}^2 \leq 2t \Re (a_j^w u_j, u_j)_{L^2}.$$

Thus, in order to prove (4.4), it is sufficient to show that for some  $j_0 \geq 0$ , all  $j \geq j_0$ , for some  $t > 0$ , there holds

$$(4.8) \quad |\Phi(-t)u_j|_{L^2} \leq |u_j|_{L^2},$$

for  $a$  and  $u$  satisfying (4.3), with  $u_j$  as in (4.4). At this stage we reformulated the Gårding inequality (4.4) into the upper bound (4.8) for the backward flow  $\Phi(-t)$  of  $a_j^w$ .

**4.3. Third step: approximation of the flow of  $\text{op}(a_j)$ .** We denote  $S_0 := e^{-ta_j}$ , and define correctors  $(S_q)_{1 \leq q \leq q_0}$  by

$$(4.9) \quad \partial_t S_q = -a_j S_q - \sum_{\substack{q_1+q_2=q \\ 0 < q_1}} a_j \diamond_{q_1} S_{q_2}, \quad S_j(0) = 0,$$

with notation  $\diamond$  introduced in (5.10).

**Lemma 4.2.** *For  $\Sigma := \sum_{0 \leq q \leq q_0-1} S_q$ , for  $q_0 := 1 + c_d \left[ \frac{\theta}{1-\theta} \right]$ , for some  $c_d > 0$  depending only on  $d$ , there holds*

$$(4.10) \quad \partial_t \Sigma^w = -a_j^w \Sigma^w + \rho(t)^w,$$

where

$$|\rho(t)^w w|_{H^s} \lesssim \sigma(t) |w|_{H^{s-q_0}}, \quad \sigma(t) := \sum_{0 \leq q \leq q_0-1} \|S_q(t)\|_{q_0-q+C(d)},$$

for some  $C(d) > 0$  depending only on  $d$ .

The reason for our choice of  $q_0$  will be apparent after Lemma 4.4.

*Proof.* By exactly the same computations as in the proof of the Approximation Lemma 2.1, we find that (4.10) holds with  $\rho = \sum_{0 \leq q \leq q_0-1} R_{q_0-q}(a_j, S_q) \in S^{-q_0}$ . The bound for  $\rho^w$  derives from (5.12).  $\square$

**Corollary 4.3.** *For some  $C, C' > 0$  depending only on  $d$ , so long as*

$$(4.11) \quad Ct2^{-jq_0} |\sigma|_{L^\infty(0,t)} < 1/2,$$

there holds the bound

$$|\Phi(-t)u_j|_{L^2} \leq |\Sigma(t)^w u_j|_{L^2} + C' t^2 2^{-jq_0} |\sigma|_{L^\infty(0,t)} \|\Sigma\|_0|_{L^\infty(0,t)} |u_j|_{L^2}.$$

*Proof.* We follow the proof of Theorem 2.2, but here we do not seek here a representation of the whole flow, only of its action on  $u_j$ . We deduce from Lemma 4.2 the representation

$$(4.12) \quad \Phi(-t)u_j = \Sigma(-t)^w u_j - \int_0^t \Sigma(-(t-t'))^w \sum_{k \geq 0} (-1)^{k+1} \rho_0^k \left( \rho(\cdot)^w u_j \right) (t') dt',$$

where  $(\rho_0 w)(t) := \int_0^t \rho(t-t')^w w(t') dt'$ . By Lemma 4.2 and a straightforward induction,

$$|\rho_0^k(\rho(\cdot)^w u_j)(t)|_{L^2} \lesssim (t2^{-jq_0} |\sigma|_{L^\infty(0,t)})^{k+1} |u_j|_{L^2},$$

using the frequency localization of  $u_j$ . From there we deduce that the sum in (4.12) converges if  $t2^{-jq_0} |\sigma|_{L^\infty(0,t)}$  is small enough, depending only on  $d$ .  $\square$

Recall that the goal is to prove (4.8). According to Corollary 4.3, it is sufficient to find  $t$  such that (4.11) holds, and also

$$(4.13) \quad C \|\Sigma(t)\|_0 + Ct^2 2^{-jq_0} |\sigma|_{L^\infty(0,t)} \|\Sigma\|_0|_{L^\infty(0,t)} \leq 1$$

for  $C > 0$  depending only on  $d$ .

**4.4. Fourth step: final estimates.** The observation time is set to

$$(4.14) \quad t_\star := j\tau_\star 2^{j\theta},$$

with  $\tau_\star > 0$  depending only on  $d$ , to be chosen large enough below.

**Lemma 4.4.** *For  $0 \leq t \leq t_\star$ , for all  $0 \leq q \leq q_0$ , all  $\alpha, \beta$ , there holds*

$$(4.15) \quad |\langle \xi \rangle^{q+|\beta|} \partial_x^\alpha \partial_\xi^\beta S_q(-t)|_{L^\infty} \leq P_j(1+t)^{q+(|\alpha|+|\beta|)/2} \exp(-t2^{-j\theta}),$$

where  $P_j$  is a polynomial in  $j$ , of degree  $2q + |\alpha| + |\beta|$ .

*Proof. First step.* On  $\{a_j < h\}$ , there holds  $|\nabla_{x,\xi} a_j| \leq 4h^{1/2}$ .

Indeed, let  $(x, \xi) \in \{a_j < h\}$ , let  $\vec{e}$  be a given unitary direction in  $\mathbb{R}^{2d}$ , and  $((x, \xi) - s_- \vec{e}, (x, \xi) + s_+ \vec{e})$  be the line segment of maximal length in  $\{a_j < h\}$  that goes through  $(x, \xi)$  and is parallel to  $\vec{e}$ . By maximality of the segment and continuity of  $a$ , the function  $\tilde{a}(s) := a_j((x, \xi) + s\vec{e})$  cannot be monotonous in  $[s_-, s_+]$ . In particular, for some  $s_0$  there holds  $\tilde{a}'(s_0) = 0$ . If  $|s_+ - s_-| \leq 2h^{1/2}$ , then this implies the bound on  $\nabla a$ , since  $|a''| \leq 1$  and  $\vec{e}$  is arbitrary. Otherwise, Taylor expansions imply

$$\tilde{a}'(s) = \frac{\tilde{a}(s + h^{1/2}) - \tilde{a}(s - h^{1/2})}{2h^{1/2}} + h^{1/2} \int_0^1 \tilde{a}''(s + h^{1/2}\sigma) - \tilde{a}''(s - h^{1/2}\sigma) d\sigma,$$

and given  $s \in (s_- + h^{1/2}, s_+ + h^{1/2})$ , we may bound  $\tilde{a}(s \pm h^{1/2})$  by  $h$ . This implies  $|\tilde{a}'(s)| \leq 3h^{1/2}$ , since  $|a''| \leq 1$ . Finally on  $(s_-, s_- + h^{1/2}]$ , we simply use another first-order Taylor expansion of  $\tilde{a}'$ , and the fact that  $|\tilde{a}'(s_- + h^{1/2})| \leq 3h^{1/2}$ . The same argument applies on  $[s_+ - h^{1/2}, s_+)$ .

*Second step.* By the Faà di Bruno formula, denoting

$$D^\gamma = \langle \xi \rangle^{|\beta|} \partial_x^\alpha \partial_\xi^\beta, \quad \text{with } \alpha + \beta = \gamma,$$

there holds

$$(4.16) \quad D^\gamma(e^{-ta_j}) = e^{-ta_j} \sum_{\substack{1 \leq k \leq |\gamma| \\ \alpha_1 + \dots + \alpha_k = \gamma}} C_{(\alpha_\ell)} t^\ell \prod_{1 \leq \ell \leq k} D^{\alpha_\ell} a_j,$$



where  $C_{(\alpha_\ell)}$  are positive constants. Let  $0 \leq k_0 \leq k$  such that  $|\alpha_\ell| = 1$  if  $\ell \leq k_0$ . Since the other indices  $\alpha_\ell$  all have length greater than two, and since there are  $k - k_0$  of them, there holds  $|\gamma| \geq k_0 + 2(k - k_0)$ . We thus obtain

$$(4.17) \quad D^\gamma(e^{-ta_j}) = \sum C_\star t^k (Da_j)^{k_0} P_\star(\partial)(D^2a), \quad 2k \leq |\gamma| + k_0, \quad k_0 \leq |\gamma|,$$

where  $(Da_j)^{k_0} = \prod_{1 \leq i, j \leq d} (\partial_{x_i} a)^{\gamma_0^{(i)}} (\langle \xi \rangle \partial_{\xi_j} a)^{\gamma_0^{(j)}}$ , for some  $\gamma_0 \in \mathbb{N}^{2d}$  such that  $|\gamma_0| = k_0$ , and  $P_\star$  is a constant-coefficient polynomial, so that  $P_\star(\partial)(D^2a)$  involves only (weighted) derivatives of  $a$  of order at least two. In (4.17), the sum runs over all possible decompositions of  $\gamma$  as in (4.16), and the  $C_\star$  are positive constants.

*Third step.* We now verify by induction that for all  $\gamma$ , all  $q \leq q_0$ ,

$$(4.18) \quad \langle \xi \rangle^q D^\gamma S_q = e^{-ta_j} \sum C_\star t^k (Da_j)^{k_0} P_\star(\partial)(D^2a),$$

with the same summation convention as in (4.17), and

$$(4.19) \quad \max(k_0, k) \leq 2q + |\gamma|, \quad k - k_0/2 \leq q + |\gamma|/2.$$

Recall that in (4.18),  $D^\gamma$  corresponds to a weighted derivative, so that the total weight in the left-hand side of (4.18) is  $\langle \xi \rangle^{q+|\beta|}$ , with  $\gamma = \alpha + \beta$ , as in (4.15).

For  $q = 1$ , there holds  $S_1 = 0$ , by (5.10). For  $q = 2$ ,  $S_2$  is a sum of terms of the form  $t^2 P_\star(D^2a_j) D^\gamma e^{-ta_j}$ , and of terms of the form  $t^3 P_\star(D^2a_j) D^\gamma ((Da_j)^2 e^{-ta_j})$ . In both cases, we verify conditions (4.18)-(4.19) directly, using the second step.

Suppose now that (4.18) holds for all  $q' \leq q - 1$ . By definition of  $S_q$  in (4.9),  $\langle \xi \rangle^q D^\gamma S_q$  is a sum of terms

$$\int_0^t D^{\gamma_1} (e^{(t-t')a_j}) D^{\gamma_2+q_1} a_j \langle \xi \rangle^{q_2} D^{\gamma_3+q_1} S_{q_2}(t') dt', \quad 0 < q_1, \quad q_1+q_2 = q, \quad |\gamma_1|+|\gamma_2|+|\gamma_3| = |\gamma|,$$

By (4.17) and the induction hypothesis, up to multiplication by  $C_\star P_\star(\partial)(D^2a_j)$  every term above is a sum of terms of the form  $e^{-ta_j} t^{1+k+k'} (Da_j)^{k_0+k'_0} D^{\gamma_2+q_1} a$ , with

$$2k \leq |\gamma_1| + k_0, \quad k_0 \leq |\gamma_1|, \quad k' \leq 2q_2 + |\gamma_3| + q_1, \quad k' - \frac{k'_0}{2} \leq q_2 + \frac{|\gamma_3|}{2} + \frac{q_1}{2}.$$

From there, we see that (4.18) holds at rank  $q$ , handling the case  $|\gamma_2| + q_1 \leq 1$  separately.

*Fourth step.* For  $0 \leq t \leq 2$ , the bound (4.15) follows from the previous step. We assume  $t \geq 2$  from now on, and use the bound on  $S_q$  given by the third step.

On  $\{a_j \geq 2^{-j\theta} + Ct^{-1} \ln t\}$ , bounding derivatives of  $a$  by 1 and using (4.18)-(4.19), we find that there holds  $\langle \xi \rangle^q |D^\gamma S_q| \leq C_q t^{2q+|\gamma|-C} e^{-t2^{-j\theta}}$ , implying (4.15) if  $C \geq 2q + |\gamma|$ .

On  $\{a_j < 2^{-j\theta} + Ct^{-1} \ln t\}$ , there holds  $|\nabla a_j| \leq 4(2^{-j\theta} + Ct^{-1} \ln t)^{1/2}$ , by the first step. On  $[0, t_\star]$ , with the limiting observation time  $t_\star$  as defined in (4.14), there holds  $2^{-j\theta} \leq Ct^{-1} \ln t$  if  $C$  is large enough (independently of  $j$ ). Hence the bound  $|\nabla a_j| \leq 4(2C)^{1/2} (t^{-1} \ln t)^{1/2}$ . Thus with (4.18)-(4.19), we find  $\langle \xi \rangle^q |D^\gamma S_q| \leq C_q t^{k-k_0/2} (\ln t)^{k_0} e^{-t2^{-j\theta}}$ , implying (4.15), since  $(\ln t)^{k_0} \leq j^{2q+|\gamma|}$ .  $\square$

By the Calderón-Vaillancourt theorem (bound (5.5) in Section 5),

$$|S_q(-t)^w|_{L^2 \rightarrow L^2} \lesssim \sup_{\alpha, \beta} |\langle \xi \rangle^{|\beta|} \partial_x^\alpha \partial_\xi^\beta S_q(-t)|_{L^\infty},$$

with  $|\alpha|, |\beta| \leq [d/2] + 1$ . We now use Lemma 4.4. Since the correctors  $S_q$ , for  $q \geq 1$ , are localized around frequencies  $\sim 2^j$ , and since  $\max_{t \geq 0} t^k e^{-t2^{-j\theta}} = C_k 2^{j\theta k}$ , we obtain

$$(4.20) \quad \begin{aligned} \max_{0 \leq t \leq t_*} |S_q(t)^w|_{L^2 \rightarrow L^2} &\lesssim P_j 2^{-jq+j\theta(q+C(d))}, \\ |S_q(t_*)^w|_{L^2 \rightarrow L^2} &\lesssim P_j 2^{-jq+j\theta(q+C(d))} e^{-\tau_* j}, \\ \max_{0 \leq t \leq t_*} \|S_q(t)\|_{q_0+q+C(d)} &\lesssim P_j 2^{j\theta(q_0+C(d))}, \end{aligned}$$

where  $P_j$  is a polynomial in  $j$ , of degree less than  $q_0 + C(d)$ , and  $t_*$  is the limiting observation time defined in (4.14). Since  $\theta < 1$ , we may sum the bounds in (4.20) over  $q$ , implying

$$\begin{aligned} \max_{0 \leq t \leq t_*} |\Sigma(t)^w|_{L^2 \rightarrow L^2} &\lesssim P_j 2^{j\theta C(d)}, \\ |\Sigma(t_*)^w|_{L^2 \rightarrow L^2} &\lesssim P_j 2^{j\theta C(d)} e^{-\tau_* j}, \\ |\sigma|_{L^\infty(0,t)} &\lesssim P_j 2^{j\theta(q_0+C(d))}. \end{aligned}$$

This shows that for  $\tau_*$  large enough the bound (4.13) holds at  $t = t_*$ . Indeed, the first term in (4.13) is

$$C \|\Sigma(t_*)\|_0 \leq C' P_j 2^{j\theta C(d)} e^{-\tau_* j} \leq 1/2,$$

if  $\tau_* > \theta C(d) \ln 2$ , and if  $j$  is large enough, depending on the degree  $q_0 + C(d)$  of  $P_j$ . And, by choice of  $q_0$  in Lemma 4.2, the second term in (4.13) is

$$C t_*^2 2^{-jq_0} |\sigma|_{L^\infty(0,t_*)} \|\Sigma\|_0 \leq C' P_j 2^{j\theta(q_0+C(d))-jq_0} \leq 1/2.$$

if  $j$  is large enough, depending only on  $\theta$  and  $d$ . This concludes the proof of Theorem 4.1.

**4.5. Remarks and extensions.** It is only in the first step of the proof of Lemma 4.4 that we use the assumption that  $a$  is scalar. There we take advantage of the fact that if  $a \in C^2$  is nonnegative, then  $|\nabla a| \lesssim |a|^{1/2}$  in a neighborhood of  $\{a = 0\}$ . This implies that the correctors  $S_q$  in the approximate solution operator do not grow in time like  $t^{2q+C(d)}$ , but only like  $t^{q+C(d)}$ . Considering that our construction of the order- $q_0$  solution operator is accurate only for  $t$  such that  $t2^{-jq_0}\sigma(t) < 1$  (this is Corollary 4.3) with  $\sigma$  growing in time like  $S_{q_0}$ , this gives the constraint  $2^{-jq_0}t^{q_0+C(d)} < 1$ , implying for the limiting observation time  $t_*$  the bound  $t_* = O(2^{j\theta})$ , with  $\theta < 1$ .

Now for matrix-valued symbols, we have no such bound on  $|\nabla a|$ . As a consequence, the correctors a priori grow like  $t^{2q+C(d)}$ . Our proof thus adapts to matrix-valued symbols, but only if we restrict to  $\theta < 1/2$ , corresponding to a gain of (just less than) half a derivative in Gårding.

Finally, we note that for operators in Weyl quantization, both the reductions to symbols of order zero and the Littlewood-Paley decomposition (4.2) generate errors that are  $O(|u|_{H^{-1}}^2)$ . Thus the analysis of Section 4.2 applies to the Fefferman-Phong inequality ([5, 1]; Theorem 2.5.10 in [9]), a refinement of Gårding with gain of two derivatives, for scalar symbols:

**Proposition 4.5.** *In order to prove the Fefferman-Phong inequality*

$$\Re(a^w u, u)_{L^2} + C|u|_{H^{(m-2)/2}}^2 \geq 0,$$

known to hold for all scalar  $a \in S^m$ , some  $C > 0$ , all  $u \in H^m$ , it is sufficient to prove that for all  $a \in S^0$  such that  $a\phi_j \geq 2^{-j}$ , the following holds: for some  $C > 0$ , for  $j$  large enough, for all  $u \in L^2$ , there holds for some  $t > 0$  the bound

$$(4.21) \quad |\Phi(-t)u_j|_{L^2}^2 \leq |u_j|_{L^2}^2 + Ct\beta_j, \quad \text{with} \quad \sum_{j \geq j_0} \beta_j \leq |u|_{H^{-1}}^2.$$

where  $\Phi$  is the flow of  $(\phi_j a)^w$ ,  $u_j = \psi_j(D)u$ , and  $(\phi_j)$  and  $(\psi_j^2)$  are two Littlewood-Paley decompositions such that  $(1 - \phi_j)\psi_j^2 \equiv 0$ .

For a proof of Proposition 4.5, it suffices to follow the reductions steps of Section 4.1 and reproduce the analysis of Section 4.2. A strong point in Proposition 4.5 is that in (4.21), the time  $t$  is allowed to be dependent of  $j$  and  $u$ . Our analysis of Sections 4.3 and 4.4 falls however short of proving (4.21); it shows that for bound (4.21) to hold at time  $t_\star = O(j2^j)$ , it would be sufficient to prove bounds in  $O(t^{q/2})$  for the correctors  $S_q$ .

## 5. APPENDIX: SYMBOLS AND OPERATORS

For  $m \in \mathbb{R}$ , the class  $S^m = S_{1,0}^m$  of classical symbols is the set of all  $a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}^{N \times N}$  such that, for all  $\alpha, \beta \in \mathbb{N}^d$ , for some  $C_{\alpha\beta} > 0$ , for all  $(x, \xi) \in \mathbb{R}^{2d}$ ,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} (1 + |\xi|^2)^{(m-|\beta|)/2}.$$

Given a symbol  $a \in S^m$ , we denote  $\|a\|_r$  the norm

$$(5.1) \quad \|a\|_r := \sup_{|\alpha|+|\beta| \leq r+2([d/2]+1)} \sup_{(x, \xi) \in \mathbb{R}^{2d}} (1 + |\xi|^2)^{(|\beta|-m)/2} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|.$$

The associated operators are, in semiclassical quantization

$$(\text{op}_\varepsilon(a)u)(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \varepsilon \xi) \hat{u}(\xi) d\xi, \quad \varepsilon > 0,$$

and in Weyl quantization

$$(5.2) \quad (a^w u)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y) \cdot \xi} a\left(\frac{x+y}{2}, \xi\right) u(y) d\xi dy.$$

When  $\varepsilon = 1$ , we denote  $\text{op}_1(a) = \text{op}(a)$ . The semiclassical Sobolev norms  $\|\cdot\|_{\varepsilon, s}$  are

$$(5.3) \quad \|u\|_{\varepsilon, s} = |(1 + |\varepsilon \xi|^2)^{s/2} \hat{u}|_{L^2(\mathbb{R}_\xi^d)}.$$

When  $\varepsilon = 1$ , the norm  $\|\cdot\|_{1, s}$  is the classical  $H^s$  norm. The Calderón-Vaillancourt theorem (see for instance [3, 7]) asserts that if  $a$  belongs to  $S^m$ , then  $\text{op}_\varepsilon(a)$  extends to a linear bounded operator  $H^m \rightarrow L^2$ , with norm controlled by  $\|a\|_0$ :

$$(5.4) \quad |\text{op}_\varepsilon(a)u|_{L^2} \lesssim \|a\|_0 \|u\|_{\varepsilon, m}, \quad \text{for all } a \in S^m, \text{ all } u \in H^m,$$

the implicit constant depending only on  $d$ . The same holds true in Weyl quantization (see for instance [2], Theorem 1.2):

$$(5.5) \quad \|a^w u\|_{L^2} \lesssim \|a\|_0 \|u\|_{L^2}, \quad \text{for all } a \in S^m, \text{ all } u \in H^m.$$

Stability by composition is expressed by the equality

$$(5.6) \quad \text{op}_\varepsilon(a_1)\text{op}_\varepsilon(a_2) = \sum_{0 \leq q \leq n} \varepsilon^q \text{op}_\varepsilon(a_1 \sharp_q a_2) + \varepsilon^{n+1} \text{op}_\varepsilon(R_{n+1}(a_1, a_2)),$$

where

$$a_1 \sharp_q a_2 = \sum_{|\alpha|=q} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a_1 \partial_x^\alpha a_2,$$

and  $R_{n+1}(a_1, a_2) \in S^{m_1+m_2-(n+1)}$  satisfies

$$(5.7) \quad |\text{op}_\varepsilon(R_{n+1}(a_1, a_2))u|_{L^2} \lesssim \|a_1\|_{n+C(d)} \|a_2\|_{n+C(d)} \|u\|_{\varepsilon, m_1+m_2-n-1},$$

with  $C(d) > 0$  depending only on  $d$ . A composition result in classical quantization is given in Theorems 1.1.5 and 1.1.20 and Lemmas 4.12 and 4.1.4 of [9]. From there (5.6)-(5.7) is easily deduced by introduction of the dilations  $(h_\varepsilon)$  such that  $(h_\varepsilon u)(x) = \varepsilon^{d/2} u(\varepsilon x)$ , and the observation that  $|h_\varepsilon u|_{H^s} = \|u\|_{\varepsilon, s}$  and  $\text{op}_\varepsilon(a) = h_\varepsilon^{-1} \text{op}(\tilde{a}) h_\varepsilon$ , with  $\tilde{a}(x, \xi) := a(\varepsilon x, \xi)$ .

Specializing to  $a_1 = (1 + |\xi|^2)^{s/2}$ , the composition result and the  $H^m \rightarrow L^2$  continuity result give continuity of  $\text{op}_\varepsilon(a)$  as an operator from  $H^{s+m}$  to  $H^s$  :

$$(5.8) \quad \|\text{op}_\varepsilon(a)u\|_{\varepsilon, s} \lesssim (\|a\|_0 + \varepsilon \|a\|_{C(d)}) \|u\|_{\varepsilon, s+m}.$$

In Weyl quantization, there holds (see for instance Section 2.1.5 in [9])

$$(5.9) \quad a_1^w a_2^w = \sum_{0 \leq k \leq n} (a_1 \diamond_k a_2)^w + R_{n+1}^w(a_1, a_2),$$

where

$$(5.10) \quad a_1 \diamond_k a_2 := \sum_{|\alpha|+|\beta|=k} \frac{(-i)^{|\alpha|}}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta a_1 \partial_x^\beta \partial_\xi^\alpha a_2,$$

and

$$(5.11) \quad |R_{n+1}^w(a_1, a_2)u|_{L^2} \lesssim \|a_1\|_{n+C(d)} \|a_2\|_{n+C(d)} \|u\|_{H^{m_1+m_2-n-1}}.$$

From (5.5) and (5.9)-(5.11) we deduce

$$(5.12) \quad |a^w u|_{H^s} \lesssim \|a\|_{C(d)} \|u\|_{H^{s+m}}, \quad a \in S^m, \quad u \in H^{s+m}, \quad s, m \in \mathbb{R}.$$

Finally, in Section 3.2, we use the pointwise bound

$$(5.13) \quad |\text{op}_\varepsilon(a)u|_{L^\infty} \lesssim \|a\|_{C(d)} \|u\|_{L^\infty} \left( 1 + |\ln \varepsilon| + \ln \left( \frac{\|u\|_{\varepsilon, d/2+m+\eta}}{\|u\|_{L^\infty}} \right) \right),$$

where  $a \in S^m$ ,  $C(d) > 0$  depends only on  $d$ ,  $\eta > 0$  is arbitrary,  $\varepsilon \in (0, 1)$ ,  $u \in H^{d/2+m+\eta}$ . The implicit constant in (5.13) depends only on  $d$  and  $\eta$ . Bound (5.13) is easily derived from estimate (B.1.1) in Appendix B of [13] by introduction of dilations and weighted norms, as mentioned above for the composition result.

## REFERENCES

- [1] J.-M. Bony, *Sur l'inégalité de Fefferman-Phong*, Séminaire: Équations aux Dérivées Partielles, 1998-1999, Exp. No. III, 16 pp., École Polytech.
- [2] A. Boulkemair,  *$L^2$  estimates for Weyl Quantization*, J. Funct. Anal. 165 (1999) 173-204.
- [3] A. Calderón, R. Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan 23 (1971) 374-378.
- [4] K.-J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations*. Graduate Texts in Mathematics, 194. Springer-Verlag, 2000. xxii+586 pp.
- [5] C. Fefferman and D. Phong, *On positivity of pseudo-differential operators*, Proc. Nat. Acad. Sci 75 (1978), 4673-4674.
- [6] L. Hörmander, *Pseudo-differential operators and non-elliptic boundary problems*, Ann. of Math. (2) 83 (1966), 129-209.
- [7] I. L. Hwang, *The  $L^2$ -boundedness of pseudodifferential operators*. Trans. Amer. Math. Soc. 302 (1987), no. 1, 55-76.
- [8] P. D. Lax, L. Nirenberg, *On stability for difference schemes: a sharp form of Gårding's inequality*, Comm. Pure Appl. Math. 19 (1966), 473-492.
- [9] N. Lerner, *Metrics on the Phase Space and Non-Selfadjoint Pseudodifferential Operators*, Pseudo-Differential Operators. Theory and Applications, 3. Birkhäuser 2010. xii+397 pp.
- [10] N. Lerner, T. Nguyen, B. Texier, *The onset of instability in first-order systems*, preprint.
- [11] Y. Lu, B. Texier, *A stability criterion for high-frequency oscillations*, [arXiv:1307.4196](https://arxiv.org/abs/1307.4196).
- [12] G. Métivier, *Remarks on the well-posedness of the nonlinear Cauchy problem*, Geometric analysis of PDE and several complex variables, Contemp. Math., vol. 368, Amer. Math. Soc., Providence, RI, 2005, pp. 337-356.
- [13] M. Taylor, *Pseudo-differential operators and nonlinear PDE*, Progress in Mathematics, 100. Birkhäuser, 1991. 213 pp.
- [14] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics, 138. American Mathematical Society, 2012. xii+431 pp.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE UMR CNRS 7586, UNIVERSITÉ PARIS-DIDEROT, AND DÉPARTEMENT DE MATHÉMATIQUES ET APPLICATIONS UMR CNRS 8553, ÉCOLE NORMALE SUPÉRIEURE

*E-mail address:* [texier@math.jussieu.fr](mailto:texier@math.jussieu.fr)